

The University of Chicago

WEAK TOPOLOGIES
OF NORMED LINEAR SPACES

A DISSERTATION SUBMITTED TO THE
FACULTY OF THE DIVISION OF THE
PHYSICAL SCIENCES IN CANDIDACY FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS
1938

By
LEONIDAS ALAOGU

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WEAK TOPOLOGIES OF NORMED LINEAR SPACES

By LEON ALAOGU

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Introduction

The concepts of weak convergence introduced by Banach [9, pp. 122, 133]¹ are of exceeding interest, but in certain respects are rather restricted. Many of the results concerned with these concepts are in general valid only for separable spaces. It is natural, then, to turn to the general theory of limits of Moore and Smith [2] for a generalization. A unified theory of weak convergence is presented in Section 1, together with some of its consequences for weak convergence of functionals. Of these the principal result is Theorem 1:3, which states that the unit sphere of the adjoint of a Banach space is bicomact in the weak topology of functionals. This is a generalization of a result given by Banach [9, p. 123] which is valid only for separable spaces.

Characterizations of spaces equivalent or isomorphic to adjoint spaces are given in Section 2, and are applied to a class of spaces considered by Dunford and Morse [13]. A result on universal spaces is given in Section 3.

The remainder of the paper is devoted to the development of theories of integration and differentiation of abstract functions. Two theories of integration are outlined in Section 4, both of which can be obtained from that of Birkhoff [11] by replacing unconditional convergence of series by its analogues for weak convergence of functionals and elements. The second of these integrals can be specialized from that of Gelfand [15] by the addition of a uniformity condition.

In Section 5 a theory of differentiation of functions with values in an adjoint space is developed. Instead of defining the derivative as the limit of the difference quotients of the function, the class of all ultimate limit points of the difference quotients in the sense of weak convergence of functionals is considered. For a function f of bounded absolute variation in the sense of Definition 5:1, it is shown that these sets are non-null almost everywhere, so that they define a multiple-valued function, the "derivative" of f . By using the first of the integrals defined in Section 4, it is shown that the fundamental theorem of the calculus holds for "derivatives" of absolute continuous functions. Gelfand's result on weak derivatives [15], and the results of Clarkson [12], Dunford and Morse [13], Gelfand [15], and Pettis [18] on strong derivatives are derived from this result. It is of interest to note that in the derivation of Clarkson's theorem, it is shown that uniformly convex spaces are regular.

¹ The numbers in brackets refer to the bibliography.

These theories of integration and differentiation are finally applied to obtain two representation theorems, one for linear functionals on the space of Bochner integrable functions, and the other for linear transformations on the space of Lebesgue integrable functions to an arbitrary Banach space.

1. Weak convergence

It has been pointed out that in the theory of normed linear spaces there arise two concepts of weak convergence different from convergence in the norm. If the space E^* is adjoint to a Banach space E , a sequence (x_n) of elements of E is said to have the *weak limit* x if $\lim_n f(x_n) = f(x)$ for every element f of E^* [9, p. 133]. Similarly, a sequence (f_n) of elements of E^* is said to be *weakly convergent* if $\lim_n f_n(x)$ exists for every element x of E [9, p. 122]. In adjoint spaces these two types of convergence are not always identical.

Banach has used these two types of convergence notably in the study of what he has called regular closure. His results in this connection are valid only for separable spaces. To remove these separability conditions, we make use of the general theory of limits of Moore and Smith [2].

The central concept of this theory is that of a *directed set*. This name is given to a class A of elements α , together with a binary relation R between the elements of A , if the relation R is transitive, and if further, for every pair α_1, α_2 of elements of A , there is a third element α such that the relations $\alpha R \alpha_1, \alpha R \alpha_2$, hold. Limits of functions (b_α) on A to the real numbers are defined in the following manner.

DEFINITION 1:1.

- (1) $\lim \sup_\alpha b_\alpha = \inf_\alpha (\sup_{\alpha_1 R \alpha} b_{\alpha_1})$;
- (2) $\lim \inf_\alpha b_\alpha = -\lim \sup_\alpha (-b_\alpha)$;
- (3) $b = \lim_\alpha b_\alpha$ if for every $\epsilon > 0$ there is an α_ϵ such that $|b - b_\alpha| < \epsilon$ if $\alpha R \alpha_\epsilon$ holds.

These limits are well-defined, and $\lim_\alpha b_\alpha$ is a linear functional; that is, $\lim_\alpha (aa_\alpha + bb_\alpha) = a \lim_\alpha a_\alpha + b \lim_\alpha b_\alpha$. We shall use the term "directed system of elements of E " for a function on a directed set A to a Banach space E , and we shall not specify the directed set whenever it is arbitrary.

DEFINITION 1:2. If G is a total subset of the space E^* adjoint to E , and if (x_α) is a directed system of elements of E , we shall say that the system is *weakly convergent on G* if $\sup_\alpha \|x_\alpha\|$ is finite, and if $\lim_\alpha f(x_\alpha)$ exists for every f of G . If for some element x of E , $f(x) = \lim_\alpha f(x_\alpha)$ for every element f of G , we shall write

$$x = w\text{-}\lim_\alpha x_\alpha \text{ on } G.$$

If G is E^* , and if the system (x_n) is a sequence, this reduces to Banach's definition of weak convergence of elements. For weak convergence of functionals, the subset G of E^{**} can be taken to be the set E_0 of all functionals $\phi_x(f) = f(x)$, where x is in E .

Another example is due to Schauder [6, p. 61]. In a space E with a basis (ξ_n) let G be the set of coefficient functionals in the development $x = \sum_n a_n(x)\xi_n$.

The sequences weakly convergent on G are then weakly convergent in Schauder's sense.

The condition that G be total in Definition 1:2 is natural; for otherwise limits would not be unique. The condition that the directed system be bounded is superfluous if the system is a sequence, and if G is E^* ; for arbitrary directed systems and arbitrary G this need not be so. For, if E is infinite-dimensional, a discontinuous, real-valued linear functional ϕ on E can be constructed by the use of a Hamel base [9, p. 231] for E . Let A be the class of all finite-dimensional linear subspaces α of E . Then A with the inclusion relation $\supset$ is a directed set. But the section of ϕ on α is continuous, and hence is the section on α of a linear functional f_α continuous on E . Thus $\phi(x) = \lim_\alpha f_\alpha(x)$ for every x ; but the system (f_α) is not bounded in view of the next theorem.

THEOREM 1:1. *If the directed system (f_α) of elements of E^* is weakly convergent on E , and if $f(x) = \lim_\alpha f_\alpha(x)$, then f is linear and continuous on E , and $\|f\| \leq \liminf_\alpha \|f_\alpha\|$.*

The linearity of f is a consequence of the linearity of the limit operation; as for the continuity of f , we see that $|f(x)| = \lim_\alpha |f_\alpha(x)| \leq \liminf_\alpha \|f_\alpha\| \cdot \|x\|$.

The weak convergence on G of Definition 1:2 defines a topology in E , in which the bounded subspaces of E are completely regular. Let K be a bounded subset of E . Then if σ is a finite subset of G , $\epsilon > 0$, and x an element of K , let

$$(1:1) \quad U_{\sigma\epsilon}(x) = [\text{all } x_1 \text{ in } K \text{ such that } |g(x - x_1)| < \epsilon \text{ for every } g \text{ in } \sigma].$$

THEOREM 1:2. *The bounded subspace K of E with neighborhood system (1:1) is a completely regular Hausdorff space of character not exceeding $\aleph_0$ times the cardinal number of G . This topology is equivalent to the topology of K defined by the directed systems of elements of K weakly convergent on G to limits in K . Equivalent neighborhood systems are obtained if G is replaced by any set having the same linear closed extension as G .*

By the *character* of a space is meant the least cardinal number of a basis of the open sets in the space.

It is readily verified that G can be replaced by its linear extension G_L in Definition 1:2 without affecting the character of the convergence of any system. The proof of the statement that G can be replaced by its metric closure G_c is similar to the proof of Banach's Theorem 1 [9, p. 133].

Let the element x of K be the weak limit on G of the directed system (x_α) of elements of a subset K_0 of K . Then for every g_i in a finite subset σ of G , and for every $\epsilon > 0$, there is an α_i such that $|g_i(x - x_\alpha)| < \epsilon$ if $\alpha R \alpha_i$ holds. But there is an α_ϵ such that $\alpha R \alpha_i$ holds for every i , and then x_α is in $U_{\sigma\epsilon}(x)$ if $\alpha R \alpha_\epsilon$ holds. Conversely, if every neighborhood $U_{\sigma\epsilon}(x)$ of the element x of K contains an element $x_{(\sigma\epsilon)}$ of K_0 , $x = w\text{-}\lim_{(\sigma\epsilon)} x_{(\sigma\epsilon)}$, where $(\sigma\epsilon)R(\sigma_1\epsilon_1)$ means that $U_{\sigma\epsilon}(x)$ is contained in $U_{\sigma_1\epsilon_1}(x)$.

We can now replace G by the set G_0 of all functionals of the form $g/M \|g\|$, with g in G ; and M can be chosen so that $|g_0(x)| \leq 1$ if x is in K and g_0 in G_0 .

Now consider the class Q of all functions ϕ on G_0 to the closed interval $(-1, 1)$. The class Q is a bicomcompact Hausdorff space with the neighborhoods

$$(1:2) \quad U_{\sigma\epsilon}(\phi) = [\text{all } \phi_1 \text{ in } Q \text{ such that } |\phi(g) - \phi_1(g)| < \epsilon \text{ for every } g \text{ of } \sigma],$$

where σ is a finite subset of G_0 , $\epsilon > 0$, and ϕ is an element of Q . This space has been studied by Tychonoff [8, p. 546], who proves these results, and shows that the character of Q is $\aleph_0$ times the cardinal number of G_0 . But, on comparing the neighborhoods (1:1) with those of (1:2), we see that K is a subspace of Q , and hence is completely regular, as is shown by Tychonoff [8, p. 545]. Also, the cardinal number of G_0 does not exceed that of G .

THEOREM 1.3. *The unit sphere K of the space E^* adjoint to E , with the neighborhoods (1:1) for $G = E$, is a bicomcompact Hausdorff space. Its character is $\aleph_0$ times the least cardinal number τ of a subset of E total for E^* .*

Banach has shown that the linear closed extension of a subset H of E total for E^* is the whole space E [9, p. 58]. Hence, by Theorem 1:2, E can be replaced by a total subset G_0 of the unit sphere of E in defining the neighborhoods (1:1). K is then a subspace of the bicomcompact space Q of Tychonoff. Moreover, K is closed in Q , so that K is bicomcompact. For, let the element f of Q be in the closure of the subspace K . Then every neighborhood $U_{\sigma\epsilon}(f)$ contains an element $f_{\sigma\epsilon}$ of K , and as in the proof of Theorem 1:2, $f(x) = \lim_{\sigma\epsilon} f_{\sigma\epsilon}(x)$ for every x of G_0 . But then by Theorem 1:2, the system is weakly convergent, and by Theorem 1:1, it has a limit in K .

Since G_0 can be chosen to have minimal cardinal number τ , the character χ of K does not exceed $\tau\aleph_0$. The local character χ_1 of K at $f = 0$ does not exceed χ ; and Alexandroff [5, pp. 269–70] has shown that from the complete system $U_{\sigma\epsilon}(0)$ of neighborhoods of $f = 0$ a complete subsystem $V_{\sigma\epsilon}$ can be chosen of minimal cardinal number χ_1 . The character χ_1 is infinite, so that we can extend the range of ϵ in the $V_{\sigma\epsilon}$ to include all positive rational numbers. Let H_1 be the logical sum of all the sets σ occurring in the $V_{\sigma\epsilon}$. Then, since χ_1 is infinite, the cardinal number of H_1 does not exceed χ_1 . Furthermore, H_1 is total. For, if the element f of K vanishes on H_1 the sequence $f_n = f$ converges weakly both to f and to zero, so that $f = 0$. Hence $\chi = \tau\aleph_0$, since $\tau\aleph_0 \leq \chi_1 \leq \chi \leq \tau\aleph_0$.

When E is separable, $\tau \leq \aleph_0$, and the space K satisfies the second countability axiom. Theorem 1:3 has been obtained with this restriction by Banach [9, pp. 123–4].

A similar restriction occurs in Banach's treatment of the regular closure. We recall here the definitions of regular closure and transfinite closure [9, pp. 115 ff.].

DEFINITION 1.3. (1) *If θ is a limit ordinal, the transfinite sequence $(f_\xi \mid \xi < \theta)$ of elements of E^* is said to have the transfinite limit f if the system (f_ξ) is bounded, and if for every element x of E ,*

$$\liminf_{\xi} f_{\xi}(x) \leq f(x) \leq \limsup_{\xi} f_{\xi}(x).$$

(2) If every bounded transfinite sequence of elements of a subset G of E^* has a transfinite limit in G , G is said to be *transfinitely closed*.

(3) The subset G of E^* is *regularly closed* if for every functional f not in G there is an element x of E such that $f(x) \neq 0$, while $g(x) = 0$ for every element g of G .

(4) The subset G of E^* is said to be *weakly closed on E* if every directed system of elements of G converging weakly on E has a limit in G .

Banach [9, pp. 121, 125] has shown that these three closure properties are equivalent for linear subspaces G of E^* if E is separable. In the next theorem we show that this equivalence holds in arbitrary Banach spaces.

THEOREM 1:4. *If the subspace G of E is linear, the following properties of G are equivalent:*

- (1) G is *transfinitely closed*,
- (2) G is *regularly closed*,
- (3) G is *weakly closed on E* .

The first implication, that (1) implies (2), has been proved by Banach [9, p. 119]. To show that (2) implies (3), let f be the weak limit on E of a directed system of elements of G . Then, if $g(x)$ vanishes for every element g of G , so does $f(x)$, so that f is in G . Finally, to prove that (3) implies (1), let (f_ξ) be a bounded transfinite sequence of elements of G . Then the sets

$$G_\eta = [\text{all } f_\xi \text{ with } \eta \leq \xi < \theta]$$

lie in a metrically closed sphere K , which is bicomact in the weak topology of Theorem 1:3. Hence the intersection of the weak closures of the sets G_η is non-null [4, p. 259]. Any element of the intersection is in G , and is a transfinite limit of the system (f_ξ) .

THEOREM 1:5. *If the functional F is linear and continuous on E^* , and if whenever the system (f_α) of elements of E^* converges weakly to f on E , it is true that $F(f) = \lim_\alpha F(f_\alpha)$, then there is an x_0 in E such that $F(f) = f(x_0)$ for every f of E^* . The converse is also true.*

The proof of this theorem is the same as that of Banach's Theorem 8 [9, p. 131], which was proved for separable spaces.

2. Two characterizations of adjoint spaces

THEOREM 2:1. *If E is a Banach space, the following two conditions on E are equivalent:*

- (1) There exists a Banach space E_1 such that E is isomorphic [9, p. 180] to E_1^* ;
- (2) There is a total subset G of E^* and a constant $M > 0$ such that for every element x of E , $\|x\| \leq M \sup_{f \in G_L} |f(x)| / \|f\|$, where G_L is the linear extension of G ; and every directed system (x_α) of elements of E weakly convergent on G has a limit in E .

If (2) holds, for every linear and continuous functional F on the linear closed extension G_{LC} of G there is an element x_0 of E such that $F(f) = f(x_0)$ for every f of G_{LC} , and such that $\|x_0\| \leq M \cdot \|F\|$.

If E is isomorphic to E_1^* , let T be the linear transformation on E to E_1^* which

is effective. Then $\|T^{-1}\|^{-1}\|x\| \leq \|Tx\| \leq \|T\|\|x\|$. Let G be the set of all functionals $f_y(x) = (Tx)(y)$ with y in E_1 . This set is total, since $(Tx)(y) = 0$ for every y if and only if $Tx = 0$, which is equivalent to $x = 0$. To obtain the inequality involving the norm in E , we observe that $|f_y(x)| = |(Tx)(y)| \leq \|y\|\|T\|\|x\|$, so that $\|f_y\| \leq \|T\|\|y\|$. Hence we see that

$$\begin{aligned} \sup_y |f_y(x)| / \|f_y\| &\geq \sup_y |(Tx)(y)| / \|T\|\|y\| \\ &= \|Tx\| / \|T\| \geq \|x\| / \|T\|\|T^{-1}\|. \end{aligned}$$

The system (x_α) converges weakly on G if and only if the system (Tx_α) converges on E_1 . But, by Theorem 1:1, this latter system has a limit g in E_1^* , and $T^{-1}g = \text{w-lim}_\alpha x_\alpha$ on G . This completes the proof of the necessity of (2).

If (2) holds for G , it also holds for the linear closed extension G_{LC} of G , by Theorem 1:2. The functionals $\phi_x(f) = f(x)$ are linear and continuous on G_{LC} and the totality of these functionals is a total linear subspace of $(G_{LC})^*$ which is weakly closed on G . Hence, by Theorem 1:4, it is the whole space $(G_{LC})^*$, so that every F in $(G_{LC})^*$ is a ϕ_x , and $\|x\| \leq M \sup_{f \text{ in } G_{LC}} |\phi_x(f)| / \|f\| = M \|\phi_x\|$. This transformation carrying x into ϕ_x is the isomorphism between E and $(G_{LC})^*$.

COROLLARY 2:1. *If in (1) of Theorem 2:1 the word "isomorphic" is replaced by "equivalent" [9, p. 180], (2) may be replaced by:*

(2') *There is a total subset G of E^* such that $\|x\| = \sup_{f \text{ in } G_L} |f(x)| / \|f\|$, and such that the unit sphere of E is bicomact in the topology of Theorem 1:2 defined by G .*

If (2') holds, E is equivalent to $(G_{LC})^$ under the transformation indicated in Theorem 2:1.*

In this case, $1 = \|T\| = \|T^{-1}\|$, and from the construction of the set G , the bicomactness follows from Theorem 1:3. To prove that (2') is sufficient, note that the transformation at the end of Theorem 2:1 is an equivalence in this case.

THEOREM 2:2. *If the Banach space E has a basis (ξ_n) such that the series $\sum_n a_n \xi_n$ converges in the norm whenever its partial sums are bounded, then E is isomorphic to $(G_{LC})^*$, where G is the set of coefficient functionals in the development $x = \sum_n a_n(x) \xi_n$.*

Spaces of this type have been considered by Dunford and Morse [13, p. 415], who showed that the space has an equivalent norm such that for every sequence (a_i) , the sequence $(\|\sum_{i=1}^n a_i \xi_i\|)$ is non-decreasing. Since G is denumerable, it suffices to show that the unit sphere of E is weakly compact in the topology defined by G . If (x_n) is a bounded sequence, the sequences $(a_i(x_n) | n)$ are bounded for each i , so that, by the diagonal process, a subsequence (x_{n_k}) can be chosen such that $a_i = \lim_k a_i(x_{n_k})$ exists for every i . The series $\sum_i a_i \xi_i$ then converges to an element x of E , and $x = \text{w-lim}_k x_{n_k}$ on G . Also

$$\|x\| = \sup_n \left\| \sum_{i=1}^n a_i(x) \xi_i \right\| = \sup_{n, f} \left| \sum_{i=1}^n a_i(x) f(\xi_i) \right| / \|f\|.$$

But $\sum_{i=1}^n a_i f(\xi_i)$ is in G_L , and $\|\sum_{i=1}^n a_i f(\xi_i)\| \leq \|f\|$, so that condition (2') of Corollary 2:1 is satisfied.

3. Universal Banach Spaces

Theorem 1:3 suggests the possibility of constructing universal Banach spaces. Let K be the bicomact space of Theorem 1:3, and let $\phi_x = (f(x) \mid f \text{ in } K)$. Then ϕ_x is continuous on K , and is linear in x . If we define $\|\phi_x\|$ to be $\sup_{f \text{ in } K} |\phi_x(f)|$, we see that $\|x\| = \|\phi_x\|$, so that this transformation is an isometry between E and a subspace of the space of continuous functions on the bicomact space K .

When K satisfies the second countability axiom, it is well-known that K is a continuous image of Cantor's ternary set $\mathfrak{N}_0$. But if the character χ of K exceeds $\mathfrak{N}_0$, such a result is not known. Stone [14] has constructed the bicomact Boolean spaces $\mathfrak{B}_\tau$ of character τ , consisting of all functions b on a range P of cardinal number τ with values zero or unity. A basis of $\mathfrak{B}_\tau$ is formed by the finite intersections of the sets

$$U_p = \{\text{all } b \text{ in } \mathfrak{B}_\tau \text{ such that } b(p) = 0\},$$

$$U'_p = \mathfrak{B}_\tau - U_p.$$

Stone has shown that a bicomact Hausdorff space of character τ is a continuous image of a closed subset of $\mathfrak{B}_\tau$. Let T be the transformation effective for K . Then the functions $\psi_x(b) = \phi_x(Tb)$ are continuous on $\mathfrak{B}$, linear in x , and $\|x\| = \sup_{b \text{ in } \mathfrak{B}} |\psi_x(b)|$. This completes the proof of the following theorem.

THEOREM 3:1. *If the least cardinal number of a total subset of the adjoint E^* of the Banach space E is τ , then E is equivalent to a linear closed subspace of the space of all continuous real-valued functions f on a bicomact Boolean space $\mathfrak{B}$ of character $\tau\mathfrak{N}_0$, where $\|f\| = \sup_{b \text{ in } \mathfrak{B}} |f(b)|$.*

When $\tau \leq \mathfrak{N}_0$, we can take $\mathfrak{B}$ to be $\mathfrak{B}_{\mathfrak{N}_0}$, as was done by Banach [9, pp. 185-7]. But if $\tau > \mathfrak{N}_0$, the space $\mathfrak{B}$ is not uniquely determined by τ . One might conjecture that Stone's theorem could be strengthened to read "every bicomact Hausdorff space of character τ is a continuous image of $\mathfrak{B}_\tau$," and then we would have a stronger Theorem 3:1.

4. Integration of abstract functions

Gelfand [15] has given a definition of an integral for abstract-valued functions T on a domain P with a measure function. If for every element f of the adjoint E^* of the range of T the real function $f[T(p)]$ is integrable in the sense of Lebesgue as defined by Fréchet [1], the function T is said to be weakly integrable. If the range of T is the adjoint E^* of a Banach space E , one might define T to be integrable if $\phi_x[T(p)]$ is integrable for every x , where, as before, $\phi_x(f) = f(x)$. It is here that one can make a construction similar to that of Birkhoff [11]. Birkhoff's construction makes use of the norm topology of the range of T ; a

similar theory using weak convergence is a natural extension. First it is necessary to have a notion of unconditional weak convergence of series. This is at hand. For use in the following definition, we observe that the class of all finite sets σ of positive integers is a directed set with the inclusion relation $\sigma_1 \supset \sigma_2$.

DEFINITION 4:1. A series $\sum_i f_i$ of elements of the space E^* adjoint to a Banach space E is said to be weakly unconditionally convergent on E if $\lim_\sigma \sum_{i \in \sigma} f_i(x)$ exists for every element x of E .

This is the analogue for weak convergence on E of Orlicz' notion of unconditional convergence in the norm [21, p. 242]. A similar definition has been given by Gelfand [16, p. 246].

THEOREM 4:1. A necessary and sufficient condition that the series $\sum_i f_i$ of elements of E^* be weakly unconditionally convergent on E is that

$$\sup_\sigma \left\| \sum_{i \in \sigma} f_i \right\| < +\infty.$$

If this be so, $w\text{-}\lim_\sigma \sum_{i \in \sigma} f_i$ exists on E .

The class of all σ is denumerable, so that, if the condition of the definition is satisfied, the limit exists semi-uniformly. Hence, by Hildebrandt's Theorem [3, p. 313], there is a σ_1 and a constant $M > 0$ such that if σ contains σ_1 , $\left\| \sum_{i \in \sigma} f_i \right\| < M$. Then for any σ , $\left\| \sum_{i \in \sigma} f_i \right\| < M + \sum_{i \in \sigma_1} \|f_i\|$ holds. Conversely, if $\sup_\sigma \left\| \sum_{i \in \sigma} f_i \right\| < +\infty$, the series $\sum_{i=1}^\infty |f_i(x)|$ is convergent for every element x of E , and $\lim_\sigma \sum_{i \in \sigma} f_i(x) = \sum_{i=1}^\infty f_i(x)$. It is then a consequence of Theorem 1:1 that $w\text{-}\lim_\sigma \sum_{i \in \sigma} f_i$ on E exists.

One can now proceed to replace unconditional convergence by weak unconditional convergence in Birkhoff's definitions and results on the summation of series of bounded subsets of E^* . At the same time the metric closure $\bar{B}$ must be replaced by the weak closure hB . This is defined to be the set of all limits of directed systems of elements of B weakly convergent on E . If B is a bounded subset of E^* , the set hB is weakly closed on E in the sense of Definition 1:3 (4). All of Birkhoff's results in this connection are valid when so amended.

The functions to be studied are multiple-valued functions on a domain P of the type described by Birkhoff to the adjoint E^* of a Banach space E . Such a function is said to be weakly summable relative to the partition $\Pi = (A_i)$ of P if the series of sets $\sum_i T(A_i)mA_i$ is weakly summable in the sense of weak convergence on E as just indicated. If T is weakly summable relative to Π , the weak integral range of T relative to Π is defined to be $J_\Pi(T) = h[Co(\sum_i T(A_i)mA_i)]$. The function is then said to be integrable over P on E if it is weakly summable relative to some partition Π , and if the integral ranges of T have only one point in common, which is denoted by $\int_P T(p) dm$. This integral is characterized in the following theorem.

THEOREM 4:2. A function T on P to E^* is integrable over P on E if and only if it is summable relative to some partition Π , and if the real functions $\phi_z[T(p)]$ are integrable over P in the sense of Lebesgue.

Thus this integral differs from the modification of Gelfand's integral indi-

cated at the beginning of this section by the additional condition of summability, which is a sort of uniformity condition. It can be shown that this condition of integrability is weaker than that of Birkhoff, and that when the latter holds, the integrals are equal.

Much use has been made of the transformation ψ on E to E^{**} defined by the relations $\psi(x) = \phi_x$, and $\phi_x(f) = f(x)$. By means of this transformation one other integral can be defined. A function T on P to the Banach space E is said to be *integrable over P on E^** if the function $\psi[T(p)]$ with values in E^{**} is integrable over P on E^* . By Theorem 4:2 this condition can be shown to be stronger than Gelfand's, and weaker than that of Birkhoff.

5. Differentiation of abstract functions

In this section we shall consider single-valued functions on a real interval to the adjoint E^* of a Banach space E , and we shall be concerned with a theory of differentiation for such functions. As in the Lebesgue theory, a notion of *bounded variation* is necessary.

DEFINITION 5:1. A single-valued function on the interval (c, d) to the Banach space E is said to be of *bounded absolute variation* if there exists a number M such that for every partition $c = a_0 \leq a_1 \leq \dots \leq a_n = d$ of the interval, it is true that

$$\sum_{i=1}^n \|f(a_i) - f(a_{i-1})\| \leq M.$$

There is associated with each type of convergence of infinite series in a Banach space E a notion of function of bounded variation. The word *absolute* used here refers to this distinction.

The derivative of a real-valued function is the limit $f'(t) = \lim_{a \rightarrow 0} [f(t+a) - f(t)]/a$. If $f'(t)$ exists, it is the only element of the set

$$\prod_{a > 0} h[\text{all } [f(t+a) - f(t)]/a \text{ with } |a| \leq d],$$

where h denotes the closure of the indicated class of numbers. Conversely, if this set contains only one element, $f'(t)$ exists and is that element.

DEFINITION 5:2. Let $f(\cdot, t)$ be a single-valued function on the interval (c, d) to the adjoint E^* of a Banach space E . If for some t on the interval there exist constants M and d_1 such that $\|f(\cdot, t+a) - f(\cdot, t)\| < M|a|$ when $|a| < d_1$, the set

$$\phi(\cdot, t) = \prod_{0 < d \leq d_1} h[\text{all } [f(\cdot, t+a) - f(\cdot, t)]/a \text{ with } |a| < d],$$

is defined to be the "*derivative*" of f at t . As before, the letter h denotes the weak closure on E of the set indicated.

If the condition of the definition is satisfied, the sets occurring therein are bounded and weakly closed on E . But these sets are bicomact in the topology of Theorem 1:3, and they form a monotone decreasing system. Hence their intersection is non-null, in consequence of a lemma on bicomact spaces which is well-known in a special case [4, p. 259].

LEMMA 5:1. *If (G_α) is a directed system of closed non-null subsets of a bicom-
pact Hausdorff space K , and if further, $G_{\alpha_1} \subset G_{\alpha_2}$ whenever $\alpha_1 R \alpha_2$ holds, then
 $\Pi_\alpha G_\alpha$ is non-null.*

If the sets have no point in common, for every point p of K there is a set $G_\alpha(p)$ not containing p , and hence a neighborhood U_p of p such that $U_p G_\alpha(p) = 0$. These neighborhoods cover K ; and since K is bicom-
pact, a finite number U_{p_i} will cover the whole space. But then if α stands in the relation R to the α_i corresponding to the points p_i , $G_\alpha U_{p_i} = 0$ for all i . This is a contra-
diction.

THEOREM 5:1. *If the function $f(\cdot, t)$ on the interval (c, d) to the space E^* is of
bounded absolute variation, it has a "derivative" $\phi(\cdot, t)$ almost everywhere, which
is integrable over (c, d) on E . Except on a set of zero measure, $\phi(\cdot, t) = \psi(\cdot, t)g(t)$,
where $\psi(\cdot, t)$ is bounded, and $g(t)$ is a Lebesgue-integrable real function.*

The proof depends on a device used by Dunford and Morse [13, p. 419].
Let $V(t) = \sup_\Pi \sum_{i=1}^n \|f(\cdot, a_i) - f(\cdot, a_{i-1})\|$, where Π ranges over all partitions
 $c = a_0 \leq a_1 \leq \dots \leq a_n = t$ of the interval (c, t) . The function $V(t)$ is the
total variation of f on the interval (c, t) , and is non-decreasing. Let $V_1(t) =$
 $V(t) + t - c$. Then V_1 is properly increasing, and $\|f(\cdot, t_1) - f(\cdot, t_2)\| \leq$
 $|V_1(t_1) - V_1(t_2)|$. The function V_1 has a derivative except on a set A of zero
measure, so that if t is not in A , there is a constant M_t such that $|V_1(t+a) -$
 $V_1(t)| \leq M_t |a|$ for every a . Consequently f has a "derivative" $\phi(\cdot, t)$ except
on A ; and since $\|\phi(\cdot, t)\| \leq V_1'(t)$, the function ϕ satisfies the condition of
summability in Theorem 4:2.

If g lies in the set $\phi(\cdot, t)$, there exists a directed system (a_α) of real numbers
such that $g = w\text{-}\lim_\alpha [f(\cdot, t + a_\alpha) - f(\cdot, t)]/a_\alpha$, and such that $\lim_\alpha a_\alpha = 0$.
Now let $f(x, t)$ denote the value of the functional $f(\cdot, t)$ at the point x of the
space E . Then $f(x, t)$ as a function of t is of bounded variation for each x , and
so has a derivative $f'(x, t)$ except on a set A_x of zero measure. It follows that
if t is not in the set $A + A_x$, $g(x) = f'(x, t)$, so that $\phi(x, t) = f'(x, t)$ except
on the set $A + A_x$ of zero measure. Hence $\phi(x, t)$ is integrable over (c, d) on E ,
since, as is well-known, the function $f'(x, t)$ is Lebesgue-integrable.

The function $V_1'(t)$ is integrable and nowhere zero, so that $\psi(\cdot, t) =$
 $\phi(\cdot, t)/V_1'(t)$ is bounded. This completes the proof of the theorem.

DEFINITION 5:3. *Let $f(t)$ be a single-valued function on the interval (c, d) to
the Banach space E . Then f is absolutely continuous if for every $\epsilon > 0$ there
exists an $\eta > 0$ such that $\sum_k \|f(a_k) - f(b_k)\| < \epsilon$ if the intervals (a_k, b_k) are
disjoint and have the sum of their lengths less than η .*

It is readily verified that if the function $f(\cdot, t)$ on the interval (c, d) to E^* is
absolutely continuous, it is of bounded absolute variation, and that the func-
tions $f(x, t)$ are absolutely continuous for every x in E . It was shown in the
proof of Theorem 5:1 that if $\phi(x, t)$ is the "derivative" of f , $\phi(x, t) = f'(x, t)$
almost everywhere. But if $f(x, t)$ is absolutely continuous, $f(x, t) - f(x, c) =$

$\int_c^t f'(x, t) dt$. This suffices to prove the following theorem.

THEOREM 5:2. *If the function $f(\cdot, t)$ on the interval (c, d) to E^* is absolutely continuous, it has a "derivative" almost everywhere which is integrable over (c, d) on E . For each x , $\phi(x, t) = f'(x, t)$ almost everywhere, and*

$$f(\cdot, t) - f(\cdot, c) = \int_c^t \phi(\cdot, t) dt.$$

We now turn to the question of when the "derivative" is single-valued almost everywhere, and when it possesses stronger integrability properties.

DEFINITION 5:4. (1) *If the subset G of the adjoint E^* of the Banach space E is total, the function $f(t)$ on the interval (c, d) to E is said to have a weak derivative $f'(t)$ on G at t if $f'(t) = \text{w-lim}_{a \rightarrow 0} [f(t + a) - f(t)]/a$ on G .*

(2) *The function $f(t)$ on the interval (c, d) to the space E has a strong derivative $f'(t)$ at t if $f'(t) = \lim_{a \rightarrow 0} [f(t + a) - f(t)]/a$ in the norm.*

A preliminary observation can be made. Suppose that the "derivative" $\phi(\cdot, t)$ of a function $f(\cdot, t)$ with values in an adjoint space E^* is single-valued at a point t . Then $\phi(\cdot, t)$ is the weak derivative of $f(\cdot, t)$ on E at t . For, if $\phi(\cdot, t)$ exists, the set of numbers $[f(x, t + a) - f(x, t)]/a$ is bounded for each element x of E , so that if $\lim_n a_n = 0$, there is a subsequence (a_{n_k}) such that $\lim_k [f(x, t + a_{n_k}) - f(x, t)]/a_{n_k}$ exists. The sets

$$h \text{ [all } [f(\cdot, t + a_{n_k}) - f(\cdot, t)]/a_{n_k} \text{ with } k \geq m]$$

have the point $\phi(\cdot, t)$ in common, so that there is a directed set (a_α) of the numbers a_{n_k} such that $\lim_\alpha a_\alpha = 0$, and such that $\phi(\cdot, t) = \text{w-lim}_\alpha [f(\cdot, t + a_\alpha) - f(\cdot, t)]/a_\alpha$ on E . Consequently $\phi(x, t) = \lim_\alpha [f(x, t + a_\alpha) - f(x, t)]/a_\alpha = \lim_k [f(x, t + a_{n_k}) - f(x, t)]/a_{n_k}$, so that $\phi(x, t) = \lim_{a \rightarrow 0} [f(x, t + a) - f(x, t)]/a$ for every element x of E . This result is used in the proof of the next theorem, which was obtained by Gelfand [15, p. 245].

THEOREM 5:3. *If the function f on the interval (c, d) to E^* is of bounded absolute variation, and if E is separable, the "derivative" $\phi(\cdot, t)$ is almost everywhere the weak derivative of f on E .*

If furthermore the space E^ is separable, the "derivative" of f is almost everywhere the strong derivative of f . This function ϕ is then Bochner integrable, and if f is absolutely continuous, $f(\cdot, t) - f(\cdot, c) = \int_c^t \phi(\cdot, t) dt$ in Bochner's sense.*

In view of what has just been shown, it will suffice for the first part to prove that $\phi(\cdot, t)$ is single-valued almost everywhere. If then E is separable, let (x_n) be a sequence everywhere dense on E . The function $\phi(x_n, t)$ is single-valued except on a set A_n of zero measure, by Theorem 5:2; hence $\phi(\cdot, t)$ is single-valued except on $\sum_n A_n$.

To prove the second part of the theorem, we first observe that if E is separable, $\|\phi(\cdot, t)\|$ is integrable. For, if (x_n) is the sequence everywhere dense on E , $\|\phi(\cdot, t)\| = \sup_n \|\phi(x_n, t)\| / \|x_n\|$ almost everywhere, so that $\|\phi(\cdot, t)\|$ is measurable. But, by Theorem 5:1, $\|\phi(\cdot, t)\|$ is bounded above by an integrable function, so that it is itself integrable. Now let $n\phi_n(\cdot, t) = \phi(\cdot, t)$

if $\|\phi(\cdot, t)\| \leq n$, and let $\phi_n(\cdot, t) = 0$ otherwise. Then $\phi(\cdot, t) = \lim_n n\phi_n(\cdot, t)$ in the norm almost everywhere, and $\phi_n(x, t)$ is measurable for every element x of E . We denote by $\phi_n^{-1}(U)$ the set of numbers t such that $\phi_n(\cdot, t)$ lies in the subset U of E^* . Then if U is a member of the neighborhood system (1:1) used in Theorem 1:3, $\phi_n^{-1}(U)$ is measurable. But if E is separable, the sphere K of that theorem satisfies the second countability axiom, so that if U is a Borel set in the topology of Theorem 1:3, $\phi_n^{-1}(U)$ is measurable. We now note that a closed subsphere K_0 of K is weakly closed, so that K_0 is a Borel set in the weak topology; and since open subspheres of K are sums of a denumerable number of closed subspheres, these too are Borel sets in the weak topology.

If E^* is separable, every subset of K which is Borel measurable in the metric topology is a Borel set in the weak topology. Hence K can be covered by a sequence of disjoint subsets (B_k) each Borel measurable in the weak topology, and each of diameter not exceeding $1/m$. Let g_k lie in B_k , and let $\phi_{nm}(t) = g_k$ if t lies in $\phi_n^{-1}(B_k)$. Then the functions ϕ_{nm} are Bochner measurable, and $\lim_m \phi_{nm}(t) = \phi_n(\cdot, t)$ in the norm, so that the ϕ_n , and hence ϕ are Bochner measurable [10, p. 263]. Since $\|\phi(\cdot, t)\|$ is integrable, $\phi(\cdot, t)$ is Bochner integrable [10, p. 265]. But Birkhoff has shown that a Bochner integrable function is integrable in his sense to the same value [11, p. 377], and it has been indicated in Section 4 that Birkhoff integrable functions are integrable over E to the same value. Consequently, $f(\cdot, t) - f(\cdot, c) = \int_c^t \phi(\cdot, t) dt$ in Bochner's sense, if f is absolutely continuous. The proof is completed on noting that indefinite Bochner integrals have their integrands as their strong derivatives almost everywhere [10, p. 269].

If f is not absolutely continuous, one can apply a criterion of Dunford and Morse [13, p. 417] to show that $g = f - \int_c^t \phi dt$ has a strong derivative vanishing almost everywhere. If $\epsilon > 0$, there a partition $\Pi = (a_0 = c \leq a_1 \leq \dots \leq a_n = d)$ such that $V(g) \leq \sum_i \|g(\cdot, a_i) - g(\cdot, a_{i-1})\| + \epsilon/2^{j+1}$. But there exist x_i of unit norm such that $\|g(\cdot, a_i) - g(\cdot, a_{i-1})\| \leq \|g(x_i, a_i) - g(x_i, a_{i-1})\| + \epsilon/n2^{j+1} \leq V[g(x_i, \cdot), E_i] + \epsilon/n2^{j+1}$, where E_i denotes the interval (a_{i-1}, a_i) . The criterion cited can be applied to the real singular functions $g(x, t)$ to yield open sets G_i with measures each not exceeding $\epsilon/n2^j$, such that $V[g(x_i, \cdot), E_i] = V[g(x_i, \cdot), G_i]$. Also, if x is of unit norm, $V[g(x, \cdot), G] \leq V[g, G]$. Hence, if $G^j = \sum_1^n G_i$, $mG^j < \epsilon/2^j$ and $V(g) \leq V[g, G^j] + \epsilon/2^j$; and if $G = \sum_1^\infty G^j$, $mG < \epsilon$, and $V(g) = V(g, G)$. Thus the criterion cited is satisfied, and g has a strong derivative equal to zero almost everywhere, so that f has a strong derivative as stated in the theorem.

Before proceeding further with the question of strong derivatives, we prove some lemmas on uniformly convex spaces. A Banach space E is said to be *uniformly convex* if for every ϵ between zero and two there is a $\delta(\epsilon) > 0$ such that $\|x + y\| \leq 2(1 - \delta(\epsilon))$ whenever $\|x\| = \|y\| = 1$, and $\|x - y\| \geq \epsilon$ [12, p. 396]. We shall prove that a uniformly convex space E is *reflexive*; that

is, if ϕ is in the second adjoint E^{**} of E , there exists an x in E such that $\phi(f) = f(x)$ for every element f of E^* .

LEMMA 5:2. *If ϕ belongs to the second adjoint E^{**} of the Banach space E , there exists a directed system (x_α) of elements of E such that $\|x_\alpha\| = \|\phi\|$, and such that $\phi(f) = \lim_\alpha f(x_\alpha)$ for every element f of E . The directed set A can be chosen to be independent of ϕ .*

This lemma is due to Goldstine [17, p. 128].

LEMMA 5:3. *If E is uniformly convex, so is E^{**} .*

Let ϕ_1, ϕ_2 be in E^{**} , and let $\|\phi_1\| = \|\phi_2\| = 1$. By Lemma 5:2, there exist two directed systems $(x_\alpha), (y_\alpha)$, such that $\|x_\alpha\| = \|y_\alpha\| = 1$, $\lim_\alpha f(x_\alpha) = \phi_1(f)$, and $\lim_\alpha f(y_\alpha) = \phi_2(f)$. Then, if $\epsilon = \|\phi_1 - \phi_2\|$, it is true that $\epsilon \leq \liminf_\alpha \|x_\alpha - y_\alpha\|$. Hence there is an α_ϵ such that $\|x_\alpha - y_\alpha\| \geq \epsilon/2$ if $\alpha R \alpha_\epsilon$ holds. But then, since $\phi_1(f) + \phi_2(f) = \lim_\alpha f(x_\alpha + y_\alpha)$, it follows that $\|\phi_1 + \phi_2\| \leq \liminf_\alpha \|x_\alpha + y_\alpha\| \leq 2(1 - \delta(\epsilon/2))$, so that E^{**} is uniformly convex.

LEMMA 5:4. *Let the subset G of the space E^* be linear and total, and such that $\|x\| = \sup_{g \in G} |g(x)| / \|g\|$. Then, if E is uniformly convex, if $x = w\text{-}\lim_\alpha x_\alpha$ on G , and if $\lim_\alpha \|x_\alpha\| = \|x\|$, it is true that $\lim_\alpha \|x - x_\alpha\| = 0$.*

For if not, there is an $\epsilon > 0$ and for each α an α_α such that $\|x - x_{\alpha_\alpha}\| \geq \epsilon$, and such that $\alpha_\alpha R \alpha$ holds. It is then readily verified that $x = w\text{-}\lim_\alpha x_{\alpha_\alpha}$ on G , so that it can be assumed that for all α $\|x - x_\alpha\| \geq \epsilon$. It will also be sufficient to assume that $\|x\| = \|x_\alpha\| = 1$. By the same method as that used to prove Theorem 1:1, it can be shown that if $x = w\text{-}\lim_\alpha x_\alpha$ on G , $\|x\| \leq \liminf_\alpha \|x_\alpha\|$. Then, since $2x = w\text{-}\lim_\alpha (x + x_\alpha)$ on G , it follows that $2 = \|2x\| \leq \liminf_\alpha \|x + x_\alpha\| \leq 2(1 - \delta(\epsilon)) < 2$, a contradiction.

THEOREM 5:4.² *If the Banach space E or its adjoint E^* is uniformly convex, then E is reflexive.*

Suppose first that E is uniformly convex, and let ϕ be in E^{**} . Then, by Lemma 5:2, there exists a system (x_α) of elements of E such that $\|x_\alpha\| = \|\phi\|$, and such that $\phi(f) = \lim_\alpha f(x_\alpha)$. Let us set $\phi_x(f) = f(x)$, $\phi(f) = F_f(\phi)$, and let G be the set of all F_f . Then, by Lemma 5:3, E^{**} , G , ϕ , and the directed system (ϕ_{x_α}) satisfy the conditions of Lemma 5:4, so that $\lim_\alpha \|\phi_{x_\alpha} - \phi\| = 0$. But, if this be so, we can choose a sequence (x_n) of the elements x_α such that $\lim_n \|\phi_{x_n} - \phi\| = 0$. Consequently, the sequence (x_n) converges to an element x of E , and $\phi = \phi_x$, so that E is reflexive.

If E^* is uniformly convex, then E^* is reflexive. But Pettis [18, p. 421] has shown that this implies that E is reflexive. It should be noted that E will be reflexive if, instead of E or E^* being uniformly convex, one of E or E^* is equivalent to a uniformly convex space.

We can now summarize the results of Clarkson [12, p. 407], Pettis [18, p. 427], and Dunford and Morse [13, p. 415] in the following theorem, which is a corollary of Theorem 5:3.

² Proved by D. Milman, *On some criteria for the regularity of spaces of type (B)*, Comptes Rendus de l'Academie des Sciences de l'URSS, vol. 20 (1938), pp. 243-246. See also B. J. Pettis, *Duke Journal of Mathematics*, vol. 5 (1939), pp. 249-253.

THEOREM 5:5. *Each of the following conditions is sufficient that a function f on an interval (c, d) to a Banach space E which is of bounded absolute variation should have a strong derivative almost everywhere:*

- (1) *Every linear, closed, separable subspace of E is reflexive;*
- (2) *E is reflexive [18, p. 427];*
- (3) *E is uniformly convex [12, p. 407];*
- (4) *E^* is uniformly convex;*
- (5) *E has a basis (ξ_n) such that the series $\sum_n a_n \xi_n$ converges in the norm whenever its partial sums are bounded [13, p. 415].*

By Theorem 5:4, if either of E or E^* is uniformly convex, then E is reflexive. If E is reflexive, then E satisfies condition (1) [18, p. 423]. Hence to prove the sufficiency of conditions (2), (3), (4), we need only prove that (1) is sufficient.

If f is of bounded absolute variation, it is readily verified that f can have at most a denumerable number of discontinuities. It follows that the values of $f(t)$ lie in a separable subspace E_0 of E , which can be assumed to be linear and closed. If (1) holds, E_0 is reflexive, and hence one can regard f as having values in E_0^{**} . But E_0^{**} is a separable space adjoint to the space E_0^* , so that, by Theorem 5:3, f has a strong derivative almost everywhere. To prove that (5) suffices, we use Theorem 2:2. E is then separable, and is isomorphic to the adjoint E_0^* of a Banach space E_0 . From Theorem 5:3 it follows that f has a strong derivative almost everywhere.

6. Representation of linear operations

In this section two representation theorems are proved. The first of these gives the form of the linear and continuous functionals on the Banach space $\mathfrak{X}$ of all Bochner integrable functions g on an interval (c, d) to a Banach space E .

The norm in $\mathfrak{X}$ is defined by the equation $\|g\| = \int_c^d \|g(t)\| dt$ [10, p. 272].

The result given reduces to the well-known representation of linear continuous functionals on the space (L) of Lebesgue integrable functions.

THEOREM 6:1. *If the functional ξ is in $\mathfrak{X}^*$, there exists a function $\phi(\cdot, t)$ on the interval (c, d) to E^* , which is bounded and integrable over (c, d) on E , and is such that $\phi[g(t), t]$ is integrable for every g of $\mathfrak{X}$, and*

$$\xi(g) = \int_c^d \phi[g(t), t] dt,$$

with

$$\|\xi\| = \text{ess sup}_t \|\phi(\cdot, t)\|$$

Let $K(t, x, r) = x$ if $t \leq r$, and let $K(t, x, r) = 0$ otherwise. Then for each r in (c, d) and x in E , $K(\cdot, x, r)$ is in $\mathfrak{X}$ and $\|K(\cdot, x, r)\| = \|x\| \cdot |r - c|$. The function $p(\cdot, r)$ defined by the equation $p(x, r) = \xi[K(\cdot, x, r)]$ has values in E^* , and $\|p(\cdot, r) - p(\cdot, s)\| \leq \|\xi\| \cdot |r - s|$. Hence $p(\cdot, t)$ is absolutely continuous, so that by Theorem 5:2 it has a "derivative" $\phi(\cdot, t)$ almost everywhere,

such that $\|\phi(\cdot, t)\| \leq \|\xi\|$ almost everywhere. For each r and x , $\phi[K(t, x, r), t]$ is integrable with respect to t , and $\xi[K(\cdot, x, r)] = \int_c^d \phi[K(t, x, r), t] dt$. It follows that if σ is a step-function in $\mathfrak{X}$, $\xi(\sigma) = \int_c^d \phi[\sigma(t), t] dt$. Also $\|\phi[g(t), t]\| \leq \|g(t)\| \cdot \|\xi\|$ almost everywhere. If g is in $\mathfrak{X}$, there exists a sequence (σ_n) of step-functions of $\mathfrak{X}$, such that $g(t) = \lim_n \sigma_n(t)$ in the norm of E except on a set A_0 of zero measure, and such that $g = \lim_n \sigma_n$ in the norm of $\mathfrak{X}$ [10, p. 272]. Let (x_m) be the sequence of values of the step-functions σ_n ; then $\phi(x_m, t)$ is single-valued and measurable except on a set A_m of zero measure, so that $\lim_n \phi[\sigma_n(t), t] = \phi[g(t), t]$ except on $\sum_{i=0}^\infty A_i$. But $\|\phi[g(t), t]\| \leq \|\xi\| \cdot \|g(t)\|$ almost everywhere, so that $\phi[g(t), t]$ is integrable. Consequently $\xi(g) = \int_c^d \phi[g(t), t] dt$, since

$$\limsup_n \int_c^d \|\phi[g(t) - \sigma_n(t), t]\| dt \leq \limsup_n \|\xi\| \cdot \|g - \sigma_n\| = 0.$$

To determine the norm of ξ , we note that $\|\xi(g)\| \leq \text{ess sup}_t \|\phi(\cdot, t)\| \cdot \|g\|$ and that $\|\phi(\cdot, t)\| \geq \|\xi\|$ almost everywhere.

THEOREM 6:2. *If T is a linear and continuous transformation on the space (L) of Lebesgue integrable functions on the interval (c, d) to the adjoint E^* of a Banach space E , there exists a bounded function $\phi(\cdot, t)$ on (c, d) to E^* integrable over (c, d) on E such that for every function g in (L)*

$$Tg = \int_c^d g(t)\phi(\cdot, t) dt,$$

with

$$\|T\| = \text{ess sup}_t \|\phi(\cdot, t)\|.$$

Let $K(t, r) = 1$ if $t \leq r$, and let $K(t, r) = 0$ otherwise. Then $K(\cdot, r)$ is in (L) for every r in (c, d) , and $\|K(\cdot, r) - K(\cdot, s)\| = |r - s|$, so that if $p(\cdot, r) = TK(\cdot, r)$, $\|p(\cdot, r) - p(\cdot, s)\| \leq \|T\| \cdot |r - s|$. Hence p is absolutely continuous, and by Theorem 5:2, it has a "derivative" $\phi(\cdot, t)$ on (c, d) to E^* integrable on E , and such that $\|\phi(\cdot, t)\| \leq \|T\|$. If g is an element of (L) , $\phi(\cdot, t)g(t)$ is weakly summable, and if x lies in E , $\phi(x, t)g(t)$ is integrable. Hence $\phi(\cdot, t)g(t)$ is integrable over (c, d) on E , and if g is a step-function, $Tg = \int_c^d g(t)\phi(\cdot, t) dt$. But step-functions are everywhere dense in (L) , so that if $f = Tg$, $f(x) = \int_c^d \phi(x, t)g(t) dt$. Hence $Tg = \int_c^d g(t)\phi(\cdot, t) dt$, and $\|f(x)\| \leq \|g\| \cdot \|x\| \text{ess sup}_t \|\phi(\cdot, t)\|$, which determines the norm of T .

COROLLARY 6:1. *If T is a linear continuous transformation on (L) to the Banach space E , there exists a bounded function $\phi(\cdot, t)$ on (c, d) to E^{**} integrable over (c, d) on E^* such that, if $x = Tg$, and $\phi_x(f) = f(x)$, then $\phi_x = \int_c^d g(t)\phi(\cdot, t) dt$, and $\|T\| = \text{ess sup}_t \|\phi(\cdot, t)\|$.*

To prove this corollary, we make use of the transformation $\phi_x(f) = f(x)$, so that T can be regarded as having values in E^{**} .

The results of Theorems 5:3 and 5:5 can be applied if the range E of T satisfies conditions which are sufficient that functions of bounded absolute variation should have strong derivatives. For such transformations the function $\phi(\cdot, t)$ always has its values in E , and the integral of the representation is a Bochner integral.

In conclusion, it should be noted that the results of Section 5 can be obtained without trouble for functions of intervals in a Euclidean space; the proofs depend on the validity of these results for numerically-valued functions.

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